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Centers of Alternative Rings

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It is shown that I. P. Shestakov's identity, $[(x, y, z)^4, r] = 0$ in an alternative algebra, holds in any alternative ring. Specifically, characteristic not 2 does not have to be assumed. It follows that $[(x, y, z)^4, r]$ is not an element in the free alternative ring whose additive order is a positive power of two.

INTRODUCTION

I. P. Shestakov [4] proved that the identity $[(x, y, z)^4, r] = 0$ holds in any alternative algebra whose additive group contains no element of order 2. In this paper we extend this result to any alternative ring by dropping the requirement that the additive group contain no element of order 2. There is much interest in alternative ring research in removing such restrictions on characteristic because it is still unknown if the free alternative ring contains elements of finite order [2, open question No. 47]. If $[(x, y, z)^4, r] \neq 0$, then the free alternative ring on four or more generators would have an element of order 2. We will show that $[(x, y, z)^4, r] = 0$. The Kleinfeld example [3] shows that the square of an associator need not be in the commutative center.

PRELIMINARIES

Throughout this paper we let A be an arbitrary alternative ring. We let x, y, z, r be arbitrary elements of A. We let $w = (x, y, z)$. The circle product "$\circ$" indicates the symmetric product: $x \circ y = xy + yx$. The bracket product indicates the skew product: $[x, y] = xy - yx$. We list some well-known results and some of their immediate consequences which will be used in the remainder of this paper. This paper was inspired by [4]. We have used the same notation. The technique of the proofs which involves shifting w^2 back and forth within the expression until we obtain $[w^2, x]$, $w \circ [x, y]$, or (w^2, x, y) is found in [4] as well. A capital A in the identity means that the

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identity holds independent of what element of A is placed there. When two capital A's appear in the same identity, any two elements of A can be substiuted for them and the identity still holds.

$$(xy, z, r) + (x, y, [z, r]) = x(y, z, r) + (x, z, r)y \tag{1}$$

$$[x, (x, z, r)] = (x, [z, x], r) \tag{2}$$

$$(x^2, z, r) = x \circ (x, z, r) = (x, x \circ z, r) \tag{3}$$

$$[x^2, z] = x \circ [x, z] \tag{4}$$

$$(xy, z, r) = y(x, z, r) + (y, z, r)x + f(x, y, z, r) \tag{5}$$

where f is the Kleinfeld function which alternates on its four arguments.

$$[y, z] \circ w = 0 \tag{6}$$

$$w[x, y] + (w, x, y) = 0 \tag{7}$$

$$[x, y]w - (w, x, y) = 0 \tag{8}$$

$$[x, [y, z]] \circ w + 4w^2 = 0 \tag{9}$$

$$[w^2, x] = [w^2, z] = 0 \tag{10}$$

$$(y, w^2, x) = (z, w^2, x) = 0 \tag{11}$$

$$(A, w^2, [x, w]) = 0 \tag{12}$$

$$(A, w^2, [y, z]) = 0 \tag{13}$$

$$f(A, A, w^2, [y, z]) = 0 \quad \text{(Kleinfeld function)} \tag{14}$$

$$([y, z], (x, w^2, A), w) = 0 \tag{15}$$

$$[w, (A, w^2, x)] = [w, (A, w^2, z)] = 0 \tag{16}$$

$$(w, w^2, A) = 0 \tag{17}$$

$$[w^4, r] = w^2 \circ [w^2, r] \tag{18}$$

Proof of Eq. (1). The Teichmüller identity $(xy, z, r) - (x, yz, r) + (x, y, zr) = x(y, z, r) + (x, y, z)r$ holds in any ring. Simply expand the terms to verify it. The linearized right Moufang identity is $-(x, y, rz) - (x, r, yz) = -(x, y, z)r - (x, r, z)y$ [1, Eq. (2.13)]. Add these together to get $(xy, z, r) + (x, y, [z, r]) = x(y, z, r) + (x, z, r)y$. Equations (2) and (3) are from [1, Eq. (2.12), Eq. (2.13), and Eq. (2.14)]. (Equation (2.14) of [1] should read $(x, yx, z) = (x, y, zx) = x(x, y, z)$. The paper has a typographical error.) Equation (4) is from [1, Eq. (2.6)]. Equation (5) is [1, Lemma 2.1]. Equations (6), (7), and (8) are [1, Lemma 2.4]. Equations (9), (10), and (11) are [4, Lemma 1]. We prove Eq. (12). $(r, w^2, [x, w]) = (r, w, w \circ [x, w])$ by Eq. (3) $= (r, w, [x, w^2])$ by Eq. (4) $= 0$ by Eq. (10). We prove Eq. (13).

$(r, w^2, [y, z]) = (r, w, w \circ [y, z])$ by Eq. (3) $=0$ by Eq. (6). Equation (14) follows from Eq. (13). We prove Eq. (15). $([y, z], (x, w^2, r), w) = ([y, z], w \circ (x, w, r), w)$ by Eq. (3) $= ([y, z], (x, w, r), w^2)$ by Eq. (3) $=0$ by Eq. (13). We prove Eq. (16). $[w, (r, w^2, x)] = [w, (w \circ r, w, x)]$ by Eq. (3) $= (w \circ r, w, [x, w])$ by Eq. (2) $= (r, w^2, [x, w])$ by Eq. (3) $=0$ by Eq. (12). The other part is proved similarly. We prove Eq. (17). $(w, w^2, r) = (w, w, w \circ r)$ by Eq. (3) $=0$ by the alternative law. Equation (18) is from Eq. (4).

LEMMA 1. $w\{(x, w^2, r)[y, z]\} + \{[y, z](x, w^2, r)\}w = 0.$

Proof. $([x, [y,z]], w^2, r) = ([x, [y,z]] \circ w, w, r)$ by Eq. (3) $= -4(w^2, w, r)$ by Eq. (9) $=0$ by Eq. (17). Expanding $0 = ([x, [y, z]], w^2, r)$ using Eq. (5), Eq. (13), and Eq. (14), we get $0 = [y, z](x, w^2, r) - (x, w^2, r)[y, z]$. Multiply this on the right by w. By Eq. (15) the association is immaterial, and by Eq. (6) and Eq. (16) we get $0 = \{[y, z](x, w^2, r) - (x, w^2, r)[y, z]\}w = [y, z](x, w^2, r)w - (x, w^2, r)[y, z]w = [y, z](x, w^2, r)w + (x, w^2, r)w[y, z] = [y, z](x, w^2, r)w + w(x, w^2, r)[y, z] = \{[y, z](x, w^2, r)\}w + w\{(x, w^2, r)[y, z]\}$.

LEMMA 2. $[w, ((x, w^2, r), y, z)] = 0.$

Proof. For $a \in A$ let $D(a) = (a, w^2, x)$. By Eq. (1) and Eq. (10), for $a, b \in A$, $D(ab) = \{D(a)\}b + a\{D(b)\}$. It follows that $D((r, y, z)) = (D(r), y, z) + (r, D(y), z) + (r, y, D(z))$. Thus $((r, y, z), w^2, x) = ((r, w^2, x), y, z) + (r, (y, w^2, x), z) + (r, y, (z, w^2, x))$. By Eq. (11) $((r, y, z), w^2, x) = ((r, w^2, x), y, z)$. Commuting both sides by w gives $[w, ((r, w^2, x), y, z)] = [w, ((r, y, z), w^2, x)] = 0$ by Eq. (16).

THEOREM. *In any alternative ring* $[(x, y, z)^4, r] = 0.$

Proof. By Eq. (7) and Eq. (8) for any $a, b \in A$, $(x, a, b)[a, b] + ((x, a, b), a, b) = 0$ and $[a, b](x, a, b) - ((x, a, b), a, b) = 0$. We linearize them. We multiply the first on the left by w, and we multiply the second on the right by w.

$$w \begin{bmatrix} (x, y, z)[w^2, r] \\ (x, y, r)[w^2, z] \\ (x, w^2, z)[y, r] \\ (x, w^2, r)[y, z] \\ ((x, y, z), w^2, r) \\ ((x, y, r), w^2, z) \\ ((x, w^2, z), y, r) \\ ((x, w^2, r), y, z) \end{bmatrix} + \begin{bmatrix} [w^2, r](x, y, z) \\ [w^2, z](x, y, r) \\ [y, r](x, w^2, z) \\ [y, z](x, w^2, r) \\ -((x, y, z), w^2, r) \\ -((x, y, r), w^2, z) \\ -((x, w^2, z), y, r) \\ -((x, w^2, r), y, z) \end{bmatrix} w = 0$$

Using the results that $[w^2, z] = (x, w^2, z) = (w, w^2, r) = ((x, y, z), w^2, r) = 0$, which are Eq. (10), Eq. (11), and Eq. (17), and after adding, we obtain the following.

$$\left.\begin{array}{l} w^2[w^2, r] + [w^2, r]w^2 \\ + w\{(x, w^2, r)[y, z]\} + \{[y, z](x, w^2, r)\}w \\ + [w, ((x, y, r), w^2, z)] \\ + [w, ((x, w^2, r), y, z)] \end{array}\right\} = 0$$

The last three lines are zero by Lemma 1, Eq. (16), and Lemma 2. Therefore $0 = w^2[w^2, r] + [w^2, r]w^2 = [w^4, r]$ by Eq. (18).

Remarks. In I. P. Shestakov's paper [4, Lemma 2] he proved for characteristic not 2 that $w(w^2, x, r) = (w^2, x, r)w = 0$. This lemma was used extensively in his subsequent arguments. We have not been able to prove it is zero without the hypothesis of characteristic not 2. If it is not zero in the free alternative ring, then it would be an element whose additive order is a power of two.

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